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Nonlinear waves in a Kelvin–Helmholtz flow

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Nonlinear waves on the interface of two incompressible inviscid fluids of different densities and arbitrary surface tension are analysed using the method of multiple scales. Third-order equations are presented for the space and time variation of the wavenumber, frequency, amplitude and phase of stable waves. A third-order expansion is also given for wavenumbers near the linear neutrally stable wavenumbers. A second-order expansion is presented for wavenumbers near the second-harmonic resonant wavenumber, for which the fundamental and its second harmonic have the same phase velocity. This expansion shows that this resonance does not lead to instabilities.

1. Introduction

In this paper, we investigate the nonlinear stability of the interface of two semi-infinite inviscid incompressible fluids moving with uniform velocities parallel to their interface. In the absence of convective, shear, and body force instabilities, the principal instability mechanism is the Kelvin–Helmholtz mechanism (cf. Chandrasekhar 1961; Chang & Russell 1965).

The essence of the Kelvin–Helmholtz mechanism, which is the subject of this paper, is that the pressure perturbation does work on the interface. The amount of work done depends on the magnitude and phase of the pressure with respect to the wave. When the relative motion of the two fluids is subsonic, the linear pressure perturbation is 180° out of phase with the surface wave, so that it pushes down at the troughs and sucks at the crests of the waves, thereby feeding energy to the disturbance in the interface. In the supersonic case, the linear pressure perturbation is in phase with the wave slope, thus transferring the maximum energy to the interface.

As the amplitude increases, the subsonic case becomes much more unstable than the supersonic case according to the nonlinear theory of Nayfeh & Saric (1971). For the case of an inviscid gas flowing parallel to a thin viscous liquid layer they found that, for a subsonic gas, stable linear modes continue to be stable while unstable linear modes continue to be unstable but with slower growth rates. For a supersonic gas, on the other hand, they found that linear modes do not grow indefinitely but become periodic waves. These nonlinear results are qualitatively confirmed by the experiments of Gater & L'Ecuyer (1969), Saric & Marshall (1971) and Gold, Otis & Schlier (1971).

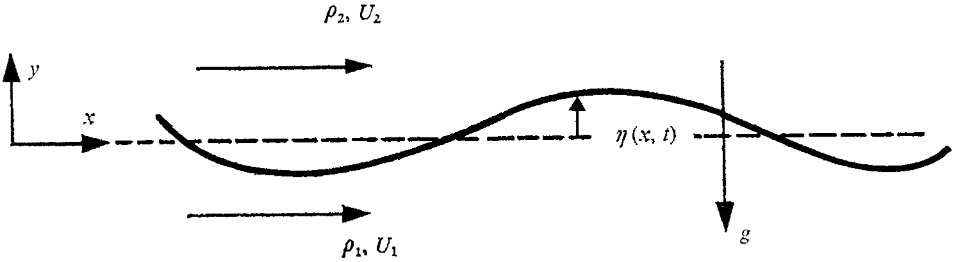

FIGURE 1. Schematic diagram of the flow.

Drazin (1970), motivated by the experiments of Thorpe (1968, 1969), conducted a nonlinear stability analysis of the interface of two incompressible inviscid fluids moving with uniform velocities parallel to their interface for the cases of (*a*) zero surface tension and (*b*) equal densities. He found that in both cases unstable linear modes correspond to finite amplitude periodic waves. Since Drazin predicted periodic waves in his case and Nayfeh & Saric predicted unstable waves in the subsonic case, we conclude that the stabilizing effect of the nonlinear motion depends on the density ratio and the value of the surface tension. One of the purposes of the present paper is to exhibit the transition from instability to stability as the density ratio increases to unity or the surface tension decreases to zero. A nonlinear stability analysis is conducted using the method of multiple scales (Nayfeh 1972*a*) for arbitrary values of the surface tension and the density ratio (see § 4). In § 5 we investigate whether the second harmonic resonance case, in which the fundamental and its second harmonic have the same linear phase velocity, leads to instability.

Maslowe & Kelly (1970) analysed, to second order, finite amplitude periodic surface waves for the case of zero surface tension. Their results do not exhibit the dependence of the wave velocity on the amplitude except near the linear neutrally stable wavenumber. A second purpose of the present paper is to use the method of multiple scales to analyse, to third order, the nonlinear motion for an arbitrary surface tension, and for waves whose frequency, wavenumber, amplitude and phase are slowly varying functions of both position and time (see § 3).

2. Problem formulation

We consider a system consisting of two semi-infinite inviscid incompressible fluids moving with uniform velocities U_1 and U_2 as shown in figure 1. We assume the initial motion to be irrotational so that the subsequent motion of the fluids is irrotational and, consequently, can be represented by potential functions.

A Cartesian co-ordinate system is introduced such that the x axis lies in the undisturbed interface while the y axis is normal to this interface and directed from fluid 1 to fluid 2 as shown in figure 1. Distances and time are made dimensionless using l and $(l/g)^{\frac{1}{2}}$, where l is a characteristic length which will be specified later and g is the body force per unit mass, which is assumed to be directed towards fluid 1.

We introduce dimensionless potential functions $\phi_j(x,y,t)$ describing the perturbed motion defined by

$$\phi_j' = (gl^3)^{\frac{1}{2}}[u_j x + \phi_j(x,y,t)], \tag{2.1}$$

where the ϕ_j' are the potential functions describing the total motion of the fluids and

$$u_j = U_j(lg)^{-\frac{1}{2}}. \tag{2.2}$$

Since the fluids are incompressible and the motion is irrotational

$$\nabla^2\phi_1 = 0, \quad -\infty < y < \eta, \tag{2.3}$$

$$\nabla^2\phi_2 = 0, \quad \eta < y < \infty, \tag{2.4}$$

where $\eta(x,t)$ is the elevation of the interface above its undisturbed position.

Equations (2.3) and (2.4) must be supplemented by boundary conditions. Away from the interface the perturbed motion vanishes, that is,

$$|\operatorname{grad}\phi_1| \to 0 \quad \text{as} \quad y \to -\infty, \tag{2.5}$$

$$|\operatorname{grad}\phi_2| \to 0 \quad \text{as} \quad y \to \infty. \tag{2.6}$$

At the interface, the kinematic condition that every particle on the interface remains on the interface leads to the two boundary conditions

$$\eta_t + u_j\eta_x + \phi_{jx}\eta_x = \phi_{jy} \quad \text{at} \quad y = \eta. \tag{2.7}$$

The balance of normal forces gives

$$\phi_{1t} + u_1\phi_{1x} + \tfrac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2) + (1-\rho)\eta = \rho[\phi_{2t} + u_2\phi_{2x} + \tfrac{1}{2}(\phi_{2x}^2 + \phi_{2y}^2)] + [1/(lk_c')^2]\eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} \quad \text{at} \quad y = \eta, \tag{2.8}$$

where ρ_1 and ρ_2 are the densities of the two fluids and

$$k_c' = (\rho_1 g/T)^{\frac{1}{2}}, \quad \rho = \rho_2/\rho_1, \tag{2.9}$$

T being the surface tension.

The problem is completed by a specification of the initial conditions. This will be done later.

3. Nonlinear dispersive waves

We now consider the propagation of a weak nonlinear wave whose wave-number, frequency, amplitude and phase are slowly varying functions of both space and time. In this case, we let $l = 1/k_c'$ so that the coefficient of the last term in (2.8) becomes unity. Following Nayfeh & Hassan (1971), we assume expansions of the form

$$\eta(x,t) = \sum_{n=1}^{3} \epsilon^n \eta_n(\theta, X_2, T_2) + O(\epsilon^4), \tag{3.1}$$

$$\phi_j(x,y,t) = \sum_{n=1}^{3} \epsilon^n \phi_{jn}(\theta, X_2, T_2, y) + O(\epsilon^4), \tag{3.2}$$

where the dimensionless parameter ϵ is of the order of the maximum steepness ratio of the wave. We assume that ϵ is small but finite, so that

$$X_2 = \epsilon^2 x, \quad T_2 = \epsilon^2 t \tag{3.3}$$

are slow position and time scales, and that θ is a rapidly rotating phase such that

$$\theta_x = k(X_2, T_2) \quad \text{(wavenumber)}, \tag{3.4}$$

$$\theta_t = -\omega(X_2, T_2) \quad \text{(frequency)}. \tag{3.5}$$

If k and ω are constants $\theta = kx - \omega t$. If θ is twice continuously differentiable, then (3.4) and (3.5) yield the compatibility relationship

$$\partial k/\partial T_2 + \partial\omega/\partial X_2 = 0, \tag{3.6}$$

or

$$\partial k/\partial T_2 + \omega'\, \partial k/\partial X_2 = 0, \tag{3.7}$$

where $\omega' = d\omega/dk$, the group velocity. Equations (3.6) and (3.7) are statements of the conservation of waves.

In terms of the new variables X_2, T_2 and θ the position and time derivatives are given by

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial\theta} + \epsilon^2\frac{\partial}{\partial X_2}, \tag{3.8}$$

$$\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial\theta} + \epsilon^2\frac{\partial}{\partial T_2}, \tag{3.9}$$

$$\frac{\partial^2}{\partial x^2} = k^2\frac{\partial^2}{\partial\theta^2} + 2\epsilon^2 k\frac{\partial^2}{\partial X_2\partial\theta} + \epsilon^2\frac{\partial k}{\partial X_2}\frac{\partial}{\partial\theta} + \epsilon^4\frac{\partial^2}{\partial X_2^2}. \tag{3.10}$$

The vertical scales $Y_1 = \epsilon y$ and $Y_2 = \epsilon^2 y$ have been ignored since our concern is not in obtaining an accurate representation of the far-field flow, but rather in obtaining the behaviour of the interface transposed to $y = 0$. In this case, as in thin airfoil problems (cf. Van Dyke 1964), the representation is correct at $y = 0$ and sufficiently accurate for $y < 1/\epsilon$. This is within the displacement of the wave.

By substituting the expansions (3.1) and (3.2) into (2.3)–(2.8), using (3.8)–(3.10) and equating coefficients of like powers of ϵ, we find that each ϕ_{1m} satisfies (2.5), while each ϕ_{2m} satisfies (2.6), and obtain equations describing η_m and ϕ_{jm}. These equations are given in the appendix.

The solution of the first-order problem is taken to be

$$\eta_1 = A(X_2, T_2)\, e^{i\theta} + \bar{A}(X_2, T_2)\, e^{-i\theta}, \tag{3.11}$$

$$\phi_{11} = i(u_1 - \omega/k)\,[A(X_2, T_2)\, e^{i\theta} - \bar{A}(X_2, T_2)\, e^{-i\theta}]\, e^{ky}, \tag{3.12}$$

$$\phi_{21} = -i(u_2 - \omega/k)\,[A(X_2, T_2)\, e^{i\theta} - \bar{A}(X_2, T_2)\, e^{-i\theta}]\, e^{-ky}, \tag{3.13}$$

where a bar denotes a complex conjugate and ω and k satisfy the dispersion relation

$$\omega = \frac{u_1 + \rho u_2}{1+\rho}k \pm \left[\frac{k^3}{1+\rho} + \frac{(1-\rho)\,k}{1+\rho} - \frac{\rho(u_1 - u_2)^2 k^2}{(1+\rho)^2}\right]^{\frac{1}{2}}. \tag{3.14}$$

The interface of the two fluids is stable or unstable according to whether ω is real or complex. Thus the flow is stable when

$$k^2 - \frac{(u_1 - u_2)^2}{(1+\rho)}\rho k + (1-\rho) > 0, \tag{3.15}$$

and neutrally stable at $k = 0$ and when the inequality of (3.15) is replaced by an equality. This is the usual result of linear stability theory (cf. Chandrasekhar 1961, § 101). In this section, we assume that (3.15) is true so that, if ω, k, and A are constants, (3.11) represents a uniform travelling wave train. In § 4, we determine expansions valid for wavenumbers near the neutrally stable wavenumbers.

The solution of the second-order problem is then

$$\eta_2 = \Lambda(A^2 e^{2i\theta} + \bar{A}^2 e^{-2i\theta}), \tag{3.16}$$

$$\phi_{12} = i(u_1 - \omega/k)(\Lambda - k)(A^2 e^{2i\theta} - \bar{A}^2 e^{-2i\theta}) e^{2ky}, \tag{3.17}$$

$$\phi_{22} = -i(u_2 - \omega/k)(\Lambda + k)(A^2 e^{2i\theta} - \bar{A}^2 e^{-2i\theta}) e^{-2ky}, \tag{3.18}$$

where
$$\Lambda = k^2 \frac{\mu_1^2 - \rho\mu_2^2}{2k(\mu_1^2 + \rho\mu_2^2) - (1-\rho) - 4k^2} \tag{3.19}$$

and
$$\mu_n = u_n - \omega/k.$$

Note that Λ and hence η_2, ϕ_{12}, and ϕ_{22} become singular when

$$4k^2 - 2k(\mu_1^2 + \mu_2^2) + 1 - \rho = 0. \tag{3.20}$$

Since
$$k^2 - k(\mu_1^2 + \rho\mu_2^2) + 1 - \rho = 0$$

from (3.14), (3.20) is satisfied when

$$\hat{k}^2 = \tfrac{1}{2}(1-\rho). \tag{3.21}$$

At this critical wavenumber, the fundamental and its second harmonic move with the same linear phase speed. Hence this critical wavenumber is the second-harmonic resonance wavenumber. Note that it is independent of the fluid velocities u_1 and u_2. An expansion valid near this resonant wavenumber is presented in § 5.

Substituting the first- and second-order solutions into (A 7)–(A 9) (see appendix), we obtain

$$k^2 \frac{\partial^2 \phi_{j3}}{\partial \theta^2} + \frac{\partial^2 \phi_{j3}}{\partial y^2} = \left[(-1)^{j+1} 2k \frac{\partial}{\partial X_2}(\mu_j A) + \mu_j A \frac{\partial k}{\partial X_2} + 2k\mu_j A \frac{\partial k}{\partial X_2} y\right] \exp\left[(-1)^{j+1} ky + i\theta\right] \quad \text{for} \quad j = 1, 2, \tag{3.22}$$

$$k\mu_j \frac{\partial \eta_3}{\partial \theta} - \frac{\partial \phi_{j3}}{\partial y} = i\mu_j k^2 [-\tfrac{1}{2}k + (-1)^{j+1} 3\Lambda] A^2 \bar{A} e^{i\theta} - \left(\frac{\partial A}{\partial T_2} + u_j \frac{\partial A}{\partial X_2}\right) e^{i\theta} + CC + NST \quad \text{for} \quad y = 0, j = 1, 2, \tag{3.23}$$

$$\begin{aligned} k\mu_1 \frac{\partial \phi_{13}}{\partial \theta} - \rho k \mu_2 \frac{\partial \phi_{23}}{\partial \theta} + (1-\rho)\eta_3 &= k^2 \frac{\partial^2 \eta_3}{\partial \theta^2} + [(\mu_1^2 - \rho\mu_2^2)\Lambda k^2 - \tfrac{5}{2}(\mu_1^2 - \rho\mu_2^2) k^3 + \tfrac{3}{2}k^4] A^2 \bar{A} e^{i\theta} \\ &\quad + i\left(2k \frac{\partial A}{\partial X_2} + A \frac{\partial k}{\partial X_2}\right) e^{i\theta} - i\left[\frac{\partial}{\partial T_2}(\mu_1 A + \rho\mu_2 A) + u_1 \frac{\partial}{\partial X_2}(\mu_1 A) \right. \\ &\quad \left. + u_2 \frac{\partial}{\partial X_2}(\mu_2 A)\right] e^{i\theta} + CC + NST \quad \text{for} \quad y = 0, \end{aligned} \tag{3.24}$$

where *CC* represents the complex conjugate and *NST* represents terms that do not produce secular terms. The particular solution of (3.22)–(3.24) contains secular terms of the form $\theta \exp(i\theta)$ which make η_3 unbounded as $\theta \to \infty$. The condition for the elimination of the secular terms can be found by requiring the inhomogeneous terms in (3.22)–(3.24) to be orthogonal to the solution of the adjoint homogeneous problem. This condition leads to the following equation:

$$2(\mu_1+\rho\mu_2)\frac{\partial A}{\partial T_2}+\left[\mu_1^2+\rho\mu_2^2-2k(1+\rho)\frac{\omega^2}{k^2}\right]\frac{\partial A}{\partial X_2}$$
$$-\left[(1+\rho)\frac{\partial}{\partial T_2}\left(\frac{\omega}{k}\right)+(1+\rho)\frac{\omega}{k}\frac{\partial}{\partial X_2}\left(\frac{\omega}{k}\right)+\frac{\partial k}{\partial X_2}\right]A = 8i(\mu_1+\rho\mu_2)JA^2\bar{A}, \quad (3.25)$$

where
$$J = \frac{4(\mu_1^2-\rho\mu_2^2)\Lambda k^2+4(\mu_1^2+\rho\mu_2^2)k^3-3k^3}{16(\mu_1+\rho\mu_2)}. \quad (3.26)$$

Using the dispersion relationship (3.14), we can rewrite (3.25) as

$$2\frac{\partial A}{\partial T_2}+2\omega'\frac{\partial A}{\partial X_2}+\omega'' A\frac{\partial k}{\partial X_2} = 8iJA^2\bar{A}. \quad (3.27)$$

This equation has the same form as those obtained by Nayfeh & Hassan (1971) for (i) waves on the interface of a subsonic gas and a liquid of finite depth, (ii) waves on the surface of a circular column of liquid and (iii) waves in a hot electron plasma.

By putting $A = \frac{1}{2}a\exp(i\beta)$, with a and β real, in (3.27) and separating real and imaginary parts, we obtain

$$\frac{\partial a^2}{\partial T_2}+\frac{\partial}{\partial X_2}(\omega' a^2) = 0, \quad (3.28)$$

$$\frac{\partial \beta}{\partial T_2}+\omega'\frac{\partial \beta}{\partial X_2} = Ja^2. \quad (3.29)$$

The surface elevation (3.1) can hence be written as

$$\eta = \epsilon a\cos(\theta+\beta)+\tfrac{1}{2}\epsilon^2 a^2\Lambda\cos 2(\theta+\beta)+O(\epsilon^3). \quad (3.30)$$

Equations (3.7) and (3.14) show that ω and k are constant along the straightline characteristics

$$dX_2/dT_2 = \omega'(k). \quad (3.31)$$

Along these characteristics (3.28) and (3.29) can be written as

$$\frac{da^2}{dT_2} = \Lambda\omega''\frac{\partial k}{\partial X_2}a^2, \quad \frac{d\beta}{dT_2} = Ja^2, \quad (3.32)$$

which can be used to compute a^2 and β.

For constant ω and k, (3.28) and (3.29) can be solved to obtain

$$a^2 = f_1(X_2-\omega' T_2), \quad (3.33)$$

$$\beta = \frac{1}{2}\frac{J}{\omega'}(X_2+\omega' T_2)f_1(X_2-\omega' T_2)+f_2(X_2-\omega' T_2), \quad (3.34)$$

where f_1 and f_2 are determined from the initial conditions. In addition, if a and β are independent of position (3.28) and (3.29) give

$$a = \text{constant}, \quad \beta = Ja^2T_2 + \beta_0. \tag{3.35}$$

Hence (3.30) becomes

$$\eta = \epsilon a \cos(kx - \omega t + J\epsilon^2 a^2 t + \beta_0) + \tfrac{1}{2}\epsilon^2 a^2 \Lambda \cos 2(kx - \omega t + J\epsilon^2 a^2 t + \beta_0) + O(\epsilon^3) \tag{3.36}$$

and the phase speed c is given by

$$c = \omega/k - (J/k)\epsilon^2 a^2. \tag{3.37}$$

This shows that the phase speed is amplitude dependent, and hence extends the results of Maslowe & Kelly (1970).

Equations (3.28) and (3.29) are valid only when the wave bandwidth $\delta k'/k' = O(\epsilon^2)$. To determine an expansion valid when $\delta k'/k' = O(\epsilon)$, we follow Benney & Newell (1967) and Stewartson & Stuart (1971) by introducing a frame of reference moving with the group velocity ω' and introducing the slow scales

$$T_2 = \epsilon^2 t, \quad \xi = \epsilon(x - \omega' t). \tag{3.38}$$

After carrying out the expansion, whose details we shall omit, we arrive at the following equation:

$$\frac{\partial A}{\partial T_2} - \tfrac{1}{2}i\omega'' \frac{\partial^2 A}{\partial \xi^2} = 4iJA^2\bar{A}, \tag{3.39}$$

where ω'' is evaluated at k_0, the centre of the wave group. This equation is similar to those obtained and discussed by Benney & Newell (1967), Taniuti & Washimi (1968), Watanabe (1969), Stewartson & Stuart (1971), Kadomtsev & Karpman (1971), DiPrima, Eckhaus & Segel (1971) and Davey (1972).

Equation (3.39) can be derived using the following short and compact procedure. Using the method of harmonic balance, or any other convenient procedure, we determine the nonlinear dispersion relationship

$$\omega = \Omega(k) - J(k)\epsilon^2 a^2 \tag{3.40}$$

for travelling waves of the form $\eta = a\exp\{i(kx - \omega t)\}$, where a, k and ω are constant. Here, $\Omega(k)$ is the right-hand side of (3.14), and $J(k)$ is given by (3.26). Next, we convert this dispersion relationship into an equivalent linear partial differential equation by replacing k and ω by $-i\partial/\partial x$ and $i\partial/\partial\omega$, respectively. Thus, we convert (3.40) into

$$\left[i\frac{\partial}{\partial t} - \Omega\left(-i\frac{\partial}{\partial x}\right) + \epsilon^2 J\left(-i\frac{\partial}{\partial x}\right)\eta\bar{\eta}\right]\eta = 0. \tag{3.41}$$

To determine an expansion for slowly varying wave trains, we apply the method of multiple scales directly to this simple equation rather than to the original equations (2.3)–(2.8). To accomplish this, we introduce the slow scales $X_1 = \mu x$ and $T_1 = \mu t$ and the rapidly rotating phase $\theta = k_0 x - \omega_0 t$, where k_0 is the centre of the wave packet and $\omega_0 = \Omega(k_0)$. Moreover, we assume that the packet bandwidth $\delta k'/k_0' = O(\mu) \geqslant O(\epsilon^2)$.

We assume that

$$\eta = A(X_1, T_1)\exp(i\theta) \tag{3.42}$$

in (3.41), where $A = \frac{1}{2}a\exp(i\beta)$ with real a and β, and obtain

$$i\mu\frac{\partial A}{\partial T_1}+i\mu\Omega'(k_0)\frac{\partial A}{\partial X_1}+\tfrac{1}{2}\mu^2\Omega''(k_0)\frac{\partial^2 A}{\partial X_1^2}+4\epsilon^2 J(k_0)A^2\bar{A} = 0. \tag{3.43}$$

If $\mu = O(\epsilon^2)$, (3.43) reduces to (3.27) if $k = k_0 =$ constant. However, if $\mu = O(\epsilon)$, we introduce a reference frame moving with the group velocity $\Omega'(k_0)$ and introduce the scales $T_2 = \epsilon T_1$ and $\xi = X_1 - \Omega'(k_0)T_1$ in (3.43) and arrive at (3.39). This procedure bears some resemblance to those reviewed by Kadomstev & Karpman (1971) and to those of Benney & Newell (1967) and Davey (1972).

4. Expansion near the neutrally stable wavenumber

We now determine an expansion valid for wavenumbers near the neutrally stable wavenumbers. In this case we let $l = 1/k'$, where k' is the wavenumber of an initial sinusoidal disturbance. The ratio $1/lk_c'$ appearing in (2.8) becomes

$$1/lk_c' = k'/k_c' = k, \tag{4.1}$$

where k is a dimensionless wavenumber. From (3.15), the dimensionless neutrally stable wavenumbers are given by

$$k_c = \frac{\chi}{2(1+\rho)} \pm \left[\frac{\chi^2}{4(1+\rho)^2}-(1-\rho)\right]^{\frac{1}{2}}, \tag{4.2}$$

where
$$\chi = \rho(U_1-U_2)^2 k_c'/g. \tag{4.3}$$

When $\rho < 1$, complete stability is obtained if $\chi^2 < 4(1+\rho)^2(1-\rho)$ or

$$(U_1-U_2)^4 < 4gT(\rho_1+\rho_2)^2(\rho_1-\rho_2)/\rho_1^2\rho_2^2,$$

in dimensional quantities (cf. Chandrasekhar 1961, chap. XI). If this condition is not satisfied, two neutrally stable wavenumbers exist for $\rho \leqslant 1$. The large one is denoted by k_T and is due to surface tension stabilization, while the small one is denoted by k_g and is due to gravity stabilization. All wavenumbers between k_T and k_g are unstable whereas those outside this range are stable. In the case where $\rho > 1$, only the neutrally stable wavenumber k_T exists because gravity is destabilizing.

The critical condition (3.15) is a function of the wavenumber, velocity difference and density ratio. Instead of solving (3.15) to determine a neutrally stable wavenumber as a function of the other variables, one can solve for a critical velocity difference or a critical density ratio. Drazin (1970) analysed the nonlinear stability of the system for the special cases: (i) $\rho = 1$ and $T \neq 0$ near the critical velocity difference, and (ii) $\rho \neq 1$ and $T = 0$ near the critical

$$\xi = (1-\rho)/(1+\rho).$$

Maslowe & Kelly (1970) investigated the existence of periodic finite amplitude waves for the case $T = 0$ near the critical wavenumber. In this paper, we determine the behaviour of the system for the general case $\rho \neq 1$ and $T \neq 1$ near the critical wavenumber by letting

$$k = k_c + \epsilon^2\sigma, \tag{4.4}$$

where $\sigma = O(1)$ is a detuning parameter.

To do this, we assume that

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(X_0, X_1, X_2, T_0, T_1, T_2) + O(\epsilon^4), \tag{4.5}$$

$$\phi_j = \sum_{n=1}^{3} \epsilon^n \phi_{jn}(X_0, X_1, X_2, T_0, T_1, T_2, y) + O(\epsilon^4), \tag{4.6}$$

where

$$T_n = \epsilon^n t, \quad X_n = \epsilon^n x.$$

We substitute these expansions into (2.3)–(2.8), equate coefficients of like powers of ϵ and obtain equations for ϕ_{jn} and η_n, where $n = 1, 2, 3$. These equations are slightly different from those in the appendix; hence we shall not present them here.

The first-order solution in this case is given by

$$\eta_1 = A(X_1, X_2, T_1, T_2)\, e^{i\theta} + \bar{A}(X_1, X_2, T_1, T_2)\, e^{-i\theta}, \tag{4.7}$$

$$\phi_{11} = i\rho\mu(Ae^{i\theta} - \bar{A}e^{-i\theta})\, e^{y}, \tag{4.8}$$

$$\phi_{21} = i\mu(Ae^{i\theta} - \bar{A}e^{-i\theta})\, e^{-y}, \tag{4.9}$$

where a bar denotes complex conjugate and

$$\theta = X_0 - cT_0, \quad c = \frac{u_1 + \rho u_2}{1+\rho}, \quad \mu = \frac{u_1 - u_2}{1+\rho}. \tag{4.10}$$

Note that $u_i = \tilde{u}_i k_c^{\frac{1}{2}}$, where $\tilde{u}_i = U_i(g/k_c')^{-\frac{1}{2}}$.

With the first-order solution known, the particular solution of the second-order problem will contain secular terms which make η_2/η_1 unbounded as T_0 or $X_0 \to \infty$ unless $\partial A/\partial X_1 = 0$. With the secular terms eliminated, the solution of the second-order problem is

$$\eta_2 = \Lambda A^2 e^{2i\theta} + CC, \tag{4.11}$$

$$\phi_{12} = -i\rho\mu(1-\Lambda)\, A^2 e^{2i\theta} e^{2y} + (\partial A/\partial T_1)\, e^{i\theta} e^{y} + CC, \tag{4.12}$$

$$\phi_{22} = i\mu(1+\Lambda)\, A^2 e^{2i\theta} e^{-2y} - (\partial A/\partial T_1)\, e^{i\theta} e^{-y} + CC, \tag{4.13}$$

where

$$\Lambda = -\rho(1-\rho)\,\mu^2/(1-\rho-2k_c^2). \tag{4.14}$$

Using the first- and second-order solutions, we find that the condition which must be satisfied for there to be no secular terms in the third-order problem is

$$(1+\rho)\frac{\partial^2 A}{\partial T_1^2} + i\,[\rho(1+\rho)\mu^2 - 2k_c^2]\frac{\partial A}{\partial X_2} + \left(2k_c - \frac{\chi}{1+\rho}\right)\sigma A = 4\Gamma A^2 \bar{A}, \tag{4.15}$$

where

$$\Gamma = \tfrac{3}{8}k_c^2 - \frac{1}{2}\frac{k_c \chi}{1+\rho} + \frac{1}{2}\frac{k_c^2 \chi^2 (1-\rho)^2}{(1+\rho)^4 (2k_c^2 - 1 + \rho)}. \tag{4.16}$$

The nonlinear Schrödinger equation (4.15) is similar in form to that obtained by Stewartson & Stuart (1971), DiPrima *et al.* (1971) and Davey (1972) if we interchange the X_1 and T_1 derivatives.

On putting $A = \frac{1}{2}a\exp(i\beta)$, with a and β real, in (4.15) and separating real and imaginary parts, we have

$$\frac{\partial^2 a}{\partial T_1^2} - a\left(\frac{\partial \beta}{\partial T_1}\right)^2 - \left(\rho\mu^2 - \frac{2k_c^2}{1+\rho}\right) a\frac{\partial \beta}{\partial X_2} + \frac{\sigma}{1+\rho}\left(2k_c - \frac{\chi}{1+\rho}\right) a = \frac{\Gamma}{1+\rho} a^3, \tag{4.17}$$

$$a\frac{\partial^2 \beta}{\partial T_1^2} + 2\frac{\partial a}{\partial T_1}\frac{\partial \beta}{\partial T_1} + \left(\rho\mu^2 - \frac{2k_c^2}{1+\rho}\right)\frac{\partial a}{\partial X_2} = 0. \tag{4.18}$$

There is no general solution known for (4.17) and (4.18). However, for a constant phase, (4.18) gives $a = a(T_1, T_2)$ and (4.17) becomes

$$\frac{\partial^2 a}{\partial T_1^2} = -\frac{\Gamma}{1+\rho}(\alpha^2 - a^2)\,a, \tag{4.19}$$

where
$$\alpha^2 = \frac{\sigma}{\Gamma}\left(2k_c - \frac{\chi}{1+\rho}\right). \tag{4.20}$$

Equation (4.19) may be integrated to give

$$\left(\frac{\partial a}{\partial T_1}\right)^2 + \frac{\Gamma}{1+\rho}(\alpha^2 a^2 - \tfrac{1}{2}a^4) = F, \tag{4.21}$$

where F is a constant determined from the initial conditions.

The behaviour of the solutions of (4.21) can be best visualized using the phase plane diagrams shown in figures 2 (a) and (b) for positive and negative Γ, respectively. For $\Gamma > 0$, figure 2 (a) shows that the motion is bounded and hence stable if $F < F_0 = \Gamma\alpha^4/2(1+\rho)$ and the trajectories intersect the $\partial a/\partial T_1$ axis. In this case, the trajectories are closed and the motion is periodic. For the special initial conditions

$$\eta(x,t) = \epsilon, \quad \partial\eta(x,t)/\partial t = 0 \quad \text{at} \quad x = ct, \tag{4.22}$$

which correspond to

$$a(0) = 1, \quad \partial a(0)/\partial T_1 = 0, \tag{4.23}$$

figure 2 (a) shows that the motion is bounded or unbounded according to whether α^2 is greater or less than unity. Hence, the special value $\alpha^2 = 1$, or

$$\frac{\sigma}{\Gamma}\left(2k_c - \frac{\chi}{1+\rho}\right) = 1, \tag{4.24}$$

separates stability from instability. Combining (4.24) with (4.4), we get the following neutrally stable wavenumber:

$$k = k_c + \epsilon^2 \frac{\Gamma}{2k_c - \chi/(1+\rho)} + O(\epsilon^3). \tag{4.25}$$

Since $2k_c - \chi/(1+\rho)$ is positive at k_T and negative at k_g according to (4.2), the nonlinear motion increases the range of unstable modes; hence, it is destabilizing.

For $\Gamma < 0$, figure 2 (b) shows that every motion is bounded. If $F \neq 0$, every motion is represented by a closed trajectory and hence is periodic. If $F = 0$, the motion, which is represented by the separatrix of figure 2 (b), tends to the origin as time increases. Therefore, if $\Gamma < 0$ the unstable linear modes correspond to finite amplitude bounded motions and, hence, the nonlinear motion is stable.

If $\rho \to 0$ while χ is kept fixed, Γ becomes

$$\Gamma = \tfrac{3}{8}k_c^2 - \tfrac{1}{2}k_c\chi + \frac{1}{2}\frac{k_c^2\chi^2}{2k_c^2 - 1} = \frac{1}{8}\frac{2k_c^4 + k_c^2 + 8}{2k_c^2 - 1} \tag{4.26}$$

in agreement with the result of Nayfeh & Saric (1971). In this case there are two critical wavenumbers according to (4.2), and $\Gamma(k_T) > 0$ while $\Gamma(k_g) < 0$ for $\chi > 3/2^{\frac{1}{2}}$ and $\Gamma(k_g) > 0$ for $\chi < 3/2^{\frac{1}{2}}$. Thus the nonlinear motion is destabilizing.

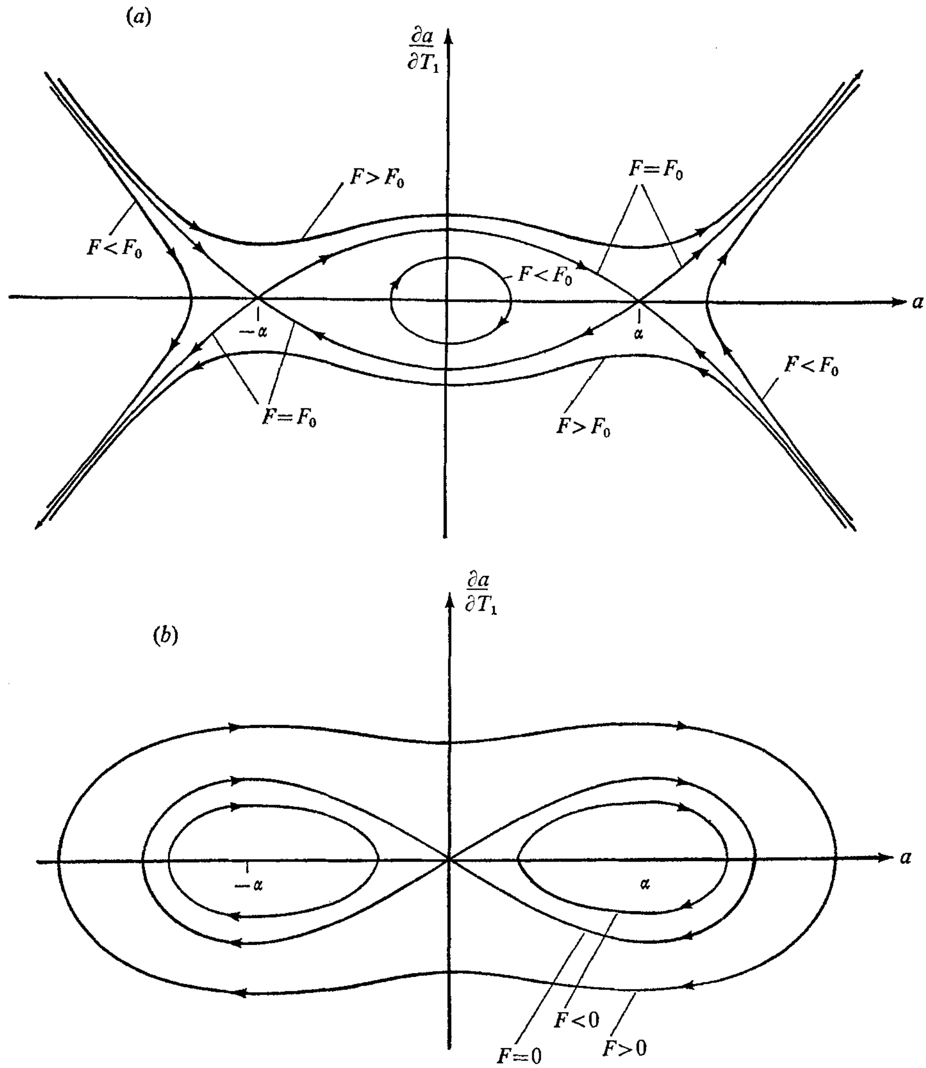

FIGURE 2. Phase plane for (*a*) $\Gamma > 0$ and (*b*) $\Gamma < 0$.

As $\rho \to 1$,
$$\Gamma \to \tfrac{3}{8}k_c^2 - \tfrac{1}{4}k_c\chi = -\tfrac{1}{32}\chi^2 < 0 \tag{4.27}$$
and, in this case, the nonlinear motion is stable. This result is qualitatively in agreement with that obtained by Drazin. Our result, however, disagrees quantitatively with Drazin's result (specifically his equation (41), which also has a misprint) because he used infinite length and zero velocity scales.

If we let $T = 0$, and use the notation and dimensionless variables of Drazin, (4.19) becomes
$$\partial^2 a/\partial t^2 + [\xi(\alpha - \alpha_c) + \tfrac{1}{2}\alpha_c^3(1+\xi^2)a^2]\,a = 0, \tag{4.28}$$
where ξ replaces ϵ, the density parameter of Drazin. The nonlinear part of this equation is the same as that obtained by Drazin, while the linear part is different from his because he perturbed about the critical ξ value ξ_c, while we perturbed about the critical wavenumber k_c, which is due to gravity stabilization in this case.

If we let a be a constant and $\beta = \beta(T_1)$ in (4.21), we get an expression for the perturbed frequency of a finite amplitude periodic wave. This frequency corresponds to that obtained by Maslowe & Kelly.

To illustrate the transition from instability near $\rho = 0$ to stability near $\rho = 1$, we show the variation of Γ with χ and ρ in figures 3 (*a*) and (*b*). For $\rho \lesssim 0{\cdot}15$, $\Gamma(k_T) > 0$ for all values of χ and the nonlinear motion is destabilizing. When $0{\cdot}15 \lesssim \rho \lesssim 0{\cdot}17$, $\Gamma(k_T)$ is positive except at intermediate values of χ. As $\chi \to \infty$, $k_g \to 0$ and $k_T \to \chi/(1+\rho)$, and hence

$$\Gamma(k_g) \to 0 \tag{4.29a}$$

and

$$\Gamma(k_T) \to \frac{\chi^2}{8(1+\rho)^2}\left[2\frac{(1-\rho)^2}{(1+\rho)^2} - 1\right]. \tag{4.29b}$$

If $\rho \geqslant 0{\cdot}1716$, (4.29*b*) shows that $\Gamma(k_T)$ is negative, and the nonlinear motion is stable as a consequence.

For small χ the situation is different because $\Gamma(k_g)$ and $\Gamma(k_T)$ are initially positive. To determine whether the nonlinear effects are stabilizing or destablizing near the minimum values of Γ, we note that

$$k_g = k_T = \frac{\chi}{2(1+\rho)} \quad \text{at} \quad \chi = \chi_c = 2(1+\rho)(1-\rho)^{\frac{1}{2}} \tag{4.30}$$

and

$$\Gamma = \tfrac{1}{2}(1-\rho)\left[(1-\rho)^2/(1+\rho)^2 - \tfrac{5}{16}\right]. \tag{4.31}$$

Equation (4.31) indicates that Γ is initially negative for $\rho \gtrsim 0{\cdot}283$. As χ increases, $\Gamma(k_T)$ remains negative but $\Gamma(k_g)$ becomes positive for some χ and returns to a negative value for $\chi > \chi_2 = (\frac{9}{8})^{\frac{1}{2}}\chi_c$, corresponding to the second-harmonic resonant wavenumber $\hat{k}$.

Although it appears that the nonlinear motion is destabilizing for wavenumbers near $\hat{k}$, it is in fact stable, as is shown in the next section. The present expansion is not valid near $\hat{k}$ because Γ is singular at $\hat{k}$.

5. Second-harmonic resonance case

In this section, we determine an expansion valid near the second-harmonic resonant wavenumber $\hat{k}$. To accomplish this, we re-introduce the dimensionless variables of § 3 and use the scales

$$T_0 = t, \quad T_1 = \epsilon t, \quad X_0 = x, \quad X_1 = \epsilon x. \tag{5.1}$$

Moreover, we assume expansions of the form

$$\eta(x,t) = \epsilon\eta_1(X_0, X_1, T_0, T_1) + \epsilon^2\eta_2(X_0, X_1, T_0, T_1) + \ldots, \tag{5.2}$$

$$\phi_j(x,y,t) = \epsilon\phi_{j1}(X_0, X_1, T_0, T_1, y) + \epsilon^2\phi_{j2}(X_0, X_1, T_0, T_1, y) + \ldots \quad (j = 1, 2). \tag{5.3}$$

Substituting (5.1)–(5.3) into (2.1)–(2.8) and equating coefficients of like powers of ϵ, we obtain problems for the determination of η_n and ϕ_{mn}. The problems are slightly different from those in the appendix, and hence we shall not present them here.

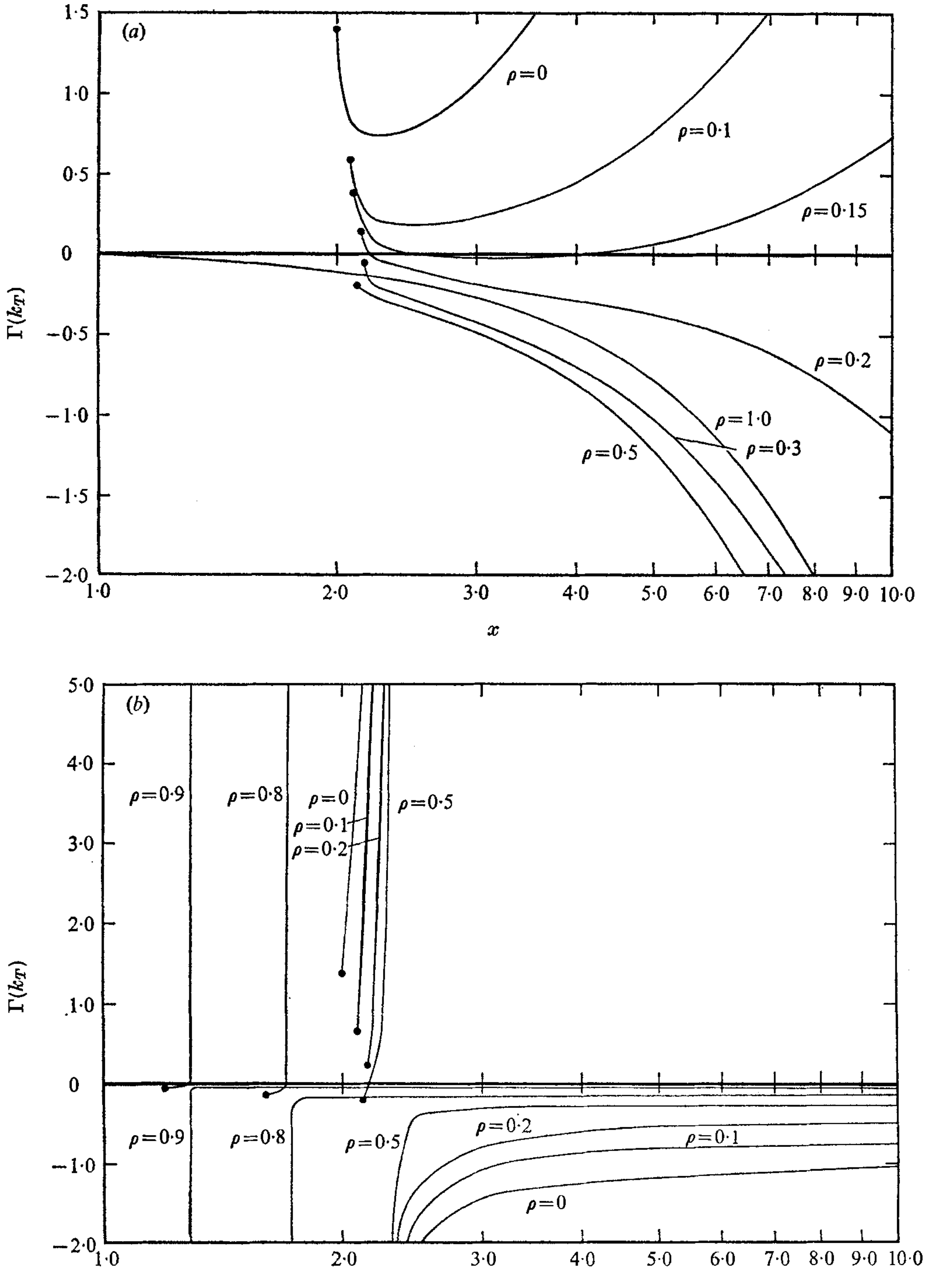

FIGURE 3. Variation of (*a*) $\Gamma(k = k_T)$ and (*b*) $\Gamma(k = k_g)$ with $\rho = \rho_2/\rho_1$ and $\chi = \rho_2(U_1 - U_2)^2/(\rho_1 g T)^{\frac{1}{2}}$.

In this case, the first-order solution is taken to contain two harmonics corresponding to

$$\eta_1 = A_1(X_1, T_1)\, e^{i\theta_1} + A_2(X_1, T_1)\, e^{i\theta_2} + CC, \tag{5.4}$$

$$\phi_{11} = i(u_1 - \omega_1/k_1) A_1 e^{i\theta_1} e^{k_1 y} + i(u_1 - \omega_2/k_2) A_2 e^{i\theta_2} e^{k_2 y} + CC, \tag{5.5 a}$$

$$\phi_{21} = -i(u_2 - \omega_1/k_1) A_1 e^{i\theta_1} e^{-k_1 y} - i(u_2 - \omega_2/k_2) A_2 e^{i\theta_2} e^{-k_2 y} + CC, \tag{5.5 b}$$

where
$$k_1 = [\tfrac{1}{2}(1-\rho)]^{\frac{1}{2}} + O(\epsilon), \quad k_2 = 2k_1 + O(\epsilon), \tag{5.6}$$

$$\theta_1 = k_1 X_0 - \omega_1 T_0, \quad \theta_2 = k_2 X_0 - \omega_2 T_0 \tag{5.7}$$

and $\omega_1 = \omega(k_1)$ and $\omega_2 = \omega(k_2)$ from (3.14). These particular values of k give $\omega_2 = 2\omega_1 + O(\epsilon)$. Therefore,

$$2\theta_1 = \theta_2 - \tilde{\alpha}(X_1, T_1), \tag{5.8}$$

where $\tilde{\alpha}$ is a detuning function which can be written as

$$\tilde{\alpha}(X_1, T_1) = \frac{k_2 - 2k_1}{\epsilon} X_1 - \frac{\omega_2 - 2\omega_1}{\epsilon} T_1. \tag{5.9}$$

By substituting (5.4)–(5.6) in the second-order problem, we obtain inhomogeneous equations and boundary conditions for the determination of η_2, ϕ_{12} and ϕ_{22}. The particular solution of the resulting second-order problem contains secular terms which make η_2/η_1 unbounded as θ_1 and $\theta_2 \to \infty$. By requiring the inhomogeneous part of this problem be orthogonal to the solution of the adjoint homogeneous problem, we obtain equations for $A_n(X_1, T_1)$. Letting

$$A_n = a_n \exp i\beta_n$$

with a_n and β_n real, we can write the surface elevation as

$$\eta = \epsilon a_1 \cos(k_1 x - \omega_1 t + \beta_1) + \epsilon a_2 \cos(k_2 x - \omega_2 t + \beta_2) + O(\epsilon^2), \tag{5.10}$$

where
$$\frac{\partial a_1}{\partial T_1} + \omega_1' \frac{\partial a_1}{\partial X_1} = -\mathcal{J} a_1 a_2 \sin\alpha, \quad \frac{\partial a_2}{\partial T_1} + \omega_2' \frac{\partial a_2}{\partial X_1} = \tfrac{1}{2}\mathcal{J} a_1^2 \sin\alpha, \tag{5.11}$$

$$\frac{\partial \beta_1}{\partial T_1} + \omega_1' \frac{\partial \beta_1}{\partial X_1} = \mathcal{J} a_2 \cos\alpha, \quad \frac{\partial \beta_2}{\partial T_1} + \omega_2' \frac{\partial \beta_2}{\partial X_1} = \tfrac{1}{2}\mathcal{J} \frac{a_1^2}{a_2} \cos\alpha \tag{5.12}$$

and
$$\alpha = \beta_2 - 2\beta_1 + \tilde{\alpha}, \quad \mathcal{J} = k_1^2 \frac{(u_1 - \omega_1/k_1)^2 - \rho(u_2 - \omega_1/k_1)^2}{(u_1 - \omega_1/k_1) + \rho(u_2 - \omega_1/k_1)}. \tag{5.13}$$

Here ω_1' and ω_2' are the group velocities of the two modes of oscillation. If $\rho = u_1 = 0$ and $k_2 = 2k_1 = 2k_c$, (5.11)–(5.13) reduce to the equations of Simmons (1969) and McGoldrick (1970) and have the same form as those obtained by Nayfeh (1972*b*).

Since there is no general solution available for (5.11) and (5.12) for general initial conditions, we investigate the temporal variation of the amplitudes and phases, following Simmons (1969), McGoldrick (1970) and Nayfeh (1972*b*). Thus, we let $\partial a_n/\partial X_1 = \partial \beta_n/\partial X_1 = 0$ and $k_2 = 2k_1$. With these assumptions, (5.11) and (5.12) have the integrals

$$a_1^2 + 2a_2^2 = E, \tag{5.14}$$

$$a_1^2 a_2 \cos\alpha - \frac{1}{\mathcal{J}} \frac{\omega_2 - 2\omega_1}{\epsilon} a_2^2 = L, \tag{5.15}$$

where E and L are constants. Equation (5.14) is a statement of the conservation of energy and shows that the motion is completely bounded.

If we consider spatial variations instead of temporal variations, the integrals of (5.11) and (5.12) are $a_1^2 + 2\omega_2' a_2^2/\omega_1' = \tilde{E}$ and $a_1^2 a_2 \cos\alpha - (\omega_2 - 2\omega_1)\omega_2' a_2^2/(\epsilon\tilde{J}) = \tilde{L}$. If ω_1' and ω_2' are positive, the treatment in this case follows that of the temporal variation case.

Letting

$$a_1^2 = E\xi, \quad a_2^2 = \tfrac{1}{2}E(1-\xi) \tag{5.16}$$

and using (5.14) and (5.15), we rewrite (5.11) as

$$\frac{2}{EJ^2}\left(\frac{\partial\xi}{\partial T_1}\right)^2 = 2(1-\xi) - \left(\frac{2}{E^3}\right)^{\frac{1}{2}}\left[L + \frac{\omega_2 - 2\omega_1}{\epsilon\tilde{J}}E(1-\xi)\right]. \tag{5.17}$$

Hence

$$\xi = a_1^2/E = \xi_3 - (\xi_3 - \xi_2)\sin^2\{\epsilon(t-t_i)\tilde{J}[\tfrac{1}{2}E(\xi_3-\xi_1)]^{\frac{1}{2}}[(\xi_3-\xi_2)/(\xi_3-\xi_1)]^{\frac{1}{2}}\}, \tag{5.18}$$

where ξ_n is a root of the cubic function on the right-hand side of (5.17), $\xi_3 > \xi_2 > \xi_1$ and t_i is the initial time. This general solution corresponds to both amplitude- and phase-modulated waves (i.e. aperiodic waves).

For pure amplitude-modulated waves (i.e. $\beta_1 = \beta_2 =$ constant), (5.12) and (5.13) demand that $\omega_2 = 2\omega_1$ and $\cos\alpha = 0$, or $\alpha = \frac{1}{2}(2n-1)\pi$. This corresponds to perfect resonance with $k_1 = [\frac{1}{2}(1-\rho)]^{\frac{1}{2}}$. The solution of (5.11) in this case is

$$a_1 = (E)^{\frac{1}{2}}\operatorname{sech}[\pm(\tfrac{1}{2}E)^{\frac{1}{2}}\epsilon\tilde{J}t + \text{constant}], \tag{5.19}$$

$$a_2 = (\tfrac{1}{2}E)^{\frac{1}{2}}\tanh[\pm(\tfrac{1}{2}E)^{\frac{1}{2}}\epsilon\tilde{J}t + \text{constant}]. \tag{5.20}$$

Therefore, as $t\to\infty$, $a_1 \to 0$ while $a_2 \to (\frac{1}{2}E)^{\frac{1}{2}}$. Thus, the steady-state motion consists of a periodic wave independent of the fundamental.

For pure phase-modulated waves (i.e. periodic waves), $\partial a_j/\partial T_1 = 0$ and hence

$$\sin\alpha = 0, \quad \alpha = n\pi. \tag{5.21}$$

On eliminating β_1 and β_2 from (5.12) and (5.13) and using $\partial\alpha/\partial T_1 = 0$, we obtain

$$a_2 = -\frac{\omega_2 - 2\omega_1}{4\epsilon\tilde{J}}\cos n\pi \pm \left[\frac{(\omega_2 - 2\omega_1)^2}{16\epsilon^2\tilde{J}^2} + \tfrac{1}{4}a_1^2\right]^{\frac{1}{2}}. \tag{5.22}$$

In this case,

$$\beta_1 = \epsilon\tilde{J}a_2\cos n\pi t + \text{constant},$$

$$\beta_2 = 2\beta_1 + (\omega_2 - 2\omega_1)t + \text{constant},$$

and the phase speed is given by

$$c = \omega_1/k_1 - \epsilon(\tilde{J}/k_1)a_2\cos n\pi. \tag{5.23}$$

Thus pure phase-modulated waves are possible near resonance and the nonlinearity adjusts the phases to yield perfect resonance.

At certain flow conditions, ω_1' is negative while ω_2' is positive, which might ead to spatial instability (Nayfeh 1972*b*).

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Appendix

Order ϵ

$$k^2\frac{\partial^2\phi_{j1}}{\partial\theta^2}+\frac{\partial^2\phi_{j1}}{\partial y^2}=0 \quad \text{for} \quad j=1,2, \tag{A 1}$$

$$k(u_j-\omega/k)\frac{\partial\eta_1}{\partial\theta}-\frac{\partial\phi_{j1}}{\partial y}=0 \quad \text{for} \quad y=0, \quad j=1,2, \tag{A 2}$$

$$k(u_1-\omega/k)\frac{\partial\phi_{11}}{\partial\theta}+(1-\rho)\eta_1=k^2\frac{\partial^2\eta_1}{\partial\theta^2}+k\rho(u_2-\omega/k)\frac{\partial\phi_{21}}{\partial\theta} \quad \text{for} \quad y=0. \tag{A 3}$$

Order ϵ^2

$$k^2\frac{\partial^2\phi_{j2}}{\partial\theta^2}+\frac{\partial^2\phi_{j2}}{\partial y^2}=0 \quad \text{for} \quad j=1,2, \tag{A 4}$$

$$k(u_j-\omega/k)\frac{\partial\eta_2}{\partial\theta}-\frac{\partial\phi_{j2}}{\partial y}=-k^2\frac{\partial\phi_{j1}}{\partial\theta}\frac{\partial\eta_1}{\partial\theta}+\eta_1\frac{\partial^2\phi_{j1}}{\partial y^2} \quad \text{for} \quad y=0,\ j=1,2. \tag{A 5}$$

$$\begin{aligned}&\sum_{j=1}^{2}(-\rho)^{j-1}k(u_j-\omega/k)\frac{\partial\phi_{j2}}{\partial\theta}+(1-\rho)\eta_2\\&=k^2\frac{\partial^2\eta_2}{\partial\theta^2}-\sum_{j=1}^{2}(-\rho)^{j-1}\left[k(u_j-\omega/k)\eta_1\frac{\partial^2\phi_{j1}}{\partial\theta\,\partial y}+\frac{k^2}{2}\left(\frac{\partial\phi_{j1}}{\partial\theta}\right)^2+\frac{1}{2}\left(\frac{\partial\phi_{j1}}{\partial y}\right)^2\right] \quad \text{for} \quad y=0.\end{aligned} \tag{A 6}$$

Order ϵ^3

$$k^2\frac{\partial^2\phi_{j3}}{\partial\theta^2}+\frac{\partial^2\phi_{j3}}{\partial y^2}=-2k\frac{\partial^2\phi_{j1}}{\partial X_2\,\partial\theta}-\frac{\partial k}{\partial X_2}\frac{\partial\phi_{j1}}{\partial\theta} \quad \text{for} \quad j=1,2, \tag{A 7}$$

$$\begin{aligned}k(u_j-\omega/k)\frac{\partial\eta_3}{\partial\theta}-\frac{\partial\phi_{j3}}{\partial y}&=-k^2\frac{\partial\eta_2}{\partial\theta}\frac{\partial\phi_{j1}}{\partial\theta}-k^2\frac{\partial\eta_1}{\partial\theta}\left[\frac{\partial\phi_{j2}}{\partial\theta}+\eta_1\frac{\partial^2\phi_{j1}}{\partial\theta\,\partial y}\right]+\eta_1\frac{\partial^2\phi_{j2}}{\partial y^2}+\eta_2\frac{\partial^2\phi_{j1}}{\partial y^2}\\&\quad+\frac{\eta_1^2}{2}\frac{\partial^3\phi_{j1}}{\partial y^3}-\frac{\partial\eta_1}{\partial T_2}-u_j\frac{\partial\eta_1}{\partial X_2} \quad \text{for} \quad y=0,\ j=1,2,\end{aligned} \tag{A 8}$$

$$\begin{aligned}&\sum_{j=1}^{2}(-\rho)^{j-1}k(u_j-\omega/k)\frac{\partial\phi_{j3}}{\partial\theta}+(1-\rho)\eta_3\\&\quad=k^2\frac{\partial^2\eta_3}{\partial\theta^2}-\sum_{j=1}^{2}(-\rho)^{j-1}\left[k(u_j-\omega/k)\left(\eta_1\frac{\partial^2\phi_{j2}}{\partial\theta\,\partial y}+\eta_2\frac{\partial^2\phi_{j1}}{\partial\theta\,\partial y}+\frac{\eta_1^2}{2}\frac{\partial^3\phi_{j1}}{\partial\theta\,\partial y^2}\right)\right.\\&\quad\left.+k^2\frac{\partial\phi_{j1}}{\partial\theta}\left(\frac{\partial\phi_{j2}}{\partial\theta}+\eta_1\frac{\partial^2\phi_{j1}}{\partial\theta\,\partial y}\right)+\frac{\partial\phi_{j1}}{\partial y}\left(\frac{\partial\phi_{j2}}{\partial y}+\eta_1\frac{\partial^2\phi_{j1}}{\partial y^2}\right)+\frac{\partial\phi_{j1}}{\partial T_2}+u_j\frac{\partial\phi_{j1}}{\partial X_2}\right]\\&\quad-\tfrac{3}{2}k^4\left(\frac{\partial\eta_1}{\partial\theta}\right)^2\frac{\partial^2\eta_1}{\partial\theta^2}+2k\frac{\partial^2\eta_1}{\partial X_2\,\partial\theta}+\frac{\partial k}{\partial X_2}\frac{\partial\eta_1}{\partial\theta} \quad \text{for} \quad y=0.\end{aligned} \tag{A 9}$$

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